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A δ-Matroid Interpretation of Spectral Graph Cuts in Graph-Structured Machine Learning Data

Abstract:
The rapid growth of graph-structured machine learning data necessitates effective and interpretable methods for graph partitioning and structural analysis. Spectral graph techniques based on the Laplacian matrix, and in particular the Fiedler vector, are widely used to approximate graph cuts in clustering, community detection, and representation learning tasks. However, classical spectral methods are predominantly algebraic and do not explicitly incorporate combinatorial feasibility constraints that frequently arise in graph-structured learning problems. This article proposes a δ-matroid interpretation of spectral graph cuts, providing a unified analytical–combinatorial framework for constrained graph partitioning. The subject of study is methods and algorithms for interpreting the Fidler vector through delta-matroid invariants. The object of the study is spectral graph cuts in graph-structured machine learning data under generalized combinatorial constraints. The study aims to formalize the correspondence between spectral invariants of the graph Laplacian and δ-matroid invariants. Vertex subsets induced by the ordering of the Fiedler vector are interpreted as feasible sets of a δ-matroid and analyzed with respect to symmetric exchange properties and valuation consistency. The proposed approach integrates Laplacian-based spectral analysis with δ-matroid-based combinatorial validation, enabling robust and interpretable graph cuts. Experimental results on synthetic and benchmark graphs demonstrate that the δ-matroid-constrained spectral cuts preserve clustering quality while improving robustness and structural interpretability, making the framework suitable for constrained clustering, graph-based learning, and explainable artificial intelligence. The results confirm that δ-matroid theory provides a natural and expressive extension of classical spectral methods, bridging continuous spectral embeddings and discrete combinatorial optimization in machine learning.

Keywords: spectral graph cut, Fiedler vector, δ-matroid, graph Laplacian, spectral clustering, constrained clustering, graph-based machine learning, combinatorial optimization.

Abbreviations:
AI is Artificial Intelligence,
DM is Delta-Matroid,
FV is Fiedler Vector,
GSL is graph structure learning,
GV is Graph Laplacian,
ML is Machine Learning,
SC is Spectral Clustering.

Introduction

Modern machine learning methods increasingly use graph models to represent complex structured data. Social networks, knowledge graphs, biological networks, recommendation systems, and multi-agent environments are naturally described as graphs, where vertices correspond to objects and edges correspond to relationships between them. One of the key tasks in analyzing such structures is graph partitioning, i.e., dividing a set of vertices into subsets with minimal disruption to internal connectivity.

Among the most effective approaches to solving this problem are spectral methods based on the eigenvalues and eigenvectors of the graph Laplacian. The Fiedler vector plays a special role, corresponding to the second smallest eigenvalue of the Laplacian and reflecting the global connectivity structure of the graph. Spectral cutting based on the Fiedler vector is widely used in clustering, semi-supervised learning, and dimensionality reduction.

At the same time, classical spectral methods have a significant limitation: they are predominantly algebraic and do not consider combinatorial constraints that naturally arise in practical machine learning problems, such as a limited number of features, structural dependencies, or permissible subset configurations. To overcome this shortcoming, it is advisable to use generalized combinatorial models.

One such model is the δ-matroid, which generalizes the classical concept of a matroid, allowing sets of admissible elements of variable power and supporting the symmetric difference operation. This paper proposes an interpretation of the spectral cut of a graph through δ-matroid invariants, which creates a theoretical basis for combining spectral methods and combinatorial constraints in machine learning.

It is necessary to establish a correspondence between:

- spectral characteristics of a graph (λ_2, Fidler vector),
- delta-matroid invariants (bases, valuations, minimal cuts).

Scientific hypothesis: the Fidler vector of a connected graph can be interpreted as an analytical relaxation of delta-matroid invariants, where the sign structure of its components corresponds to the minimal cuts of the delta-matroid, and the value of the second eigenvalue of the Laplacian correlates with the valuations of the bases.

The subject of study is methods and algorithms for interpreting the Fidler vector through delta-matroid invariants.

The object of study is graph structures and their spectral properties.

The study aims to develop and justify an approach to interpreting Fidler's vector through delta-matroid invariants of a graph.

To achieve this purpose, the following tasks are set:

- analyze current research on spectral graph theory and delta-matroids;
- formulate a scientific hypothesis regarding the correspondence between the Fidler vector and delta-matroid invariants;
- construct a formal mathematical scheme for testing the hypothesis;
- develop an experimental testing methodology;
- interpret the expected results from a theoretical and practical viewpoint.

The results of this study are intended for a broad interdisciplinary audience working at the intersection of graph theory, theoretical computer science, and machine learning. First, the research addresses specialists in spectral graph theory and combinatorial optimization, for whom the proposed interpretation of Fiedler's vector through delta-matroid invariants offers a novel theoretical extension that connects algebraic and combinatorial perspectives on graph partitioning.

Second, the study is relevant to researchers and practitioners in machine learning and data science who apply spectral clustering and graph-based methods to real-world datasets subject to complex structural or domain-specific constraints. For this audience, the framework provides a principled way to incorporate feasibility and admissibility considerations into spectral pipelines, going beyond standard unconstrained or pairwise-constrained clustering approaches.

Third, the results are addressed to algorithm designers and applied mathematicians interested in constraint-aware optimization and scalable graph algorithms. The integration of delta-matroid theory with spectral methods highlights new directions for developing efficient algorithms that balance continuous relaxations with discrete feasibility structures.

Finally, the study may be of interest to graduate students and early-career researchers seeking a rigorous yet extensible framework for combining spectral methods with advanced combinatorial models. By articulating both theoretical foundations and methodological implications, the research aims to serve as a reference point for further investigations into constraint-aware graph analysis and structured machine learning.

Materials and Methods

The study forms a unified theoretical and algorithmic framework that combines spectral graph analysis and delta-matroid theory, and opens up prospects for further fundamental and applied research.

Theoretical verification scheme. Theorem: the existence of a mapping between the Fidler vector and delta-matroid invariants. Lemmas: correspondence of minimal cuts, correlation of λ_2 and valuations, stability of the spectrum. Experiment: numerical verification on different classes of graphs (*Figure 1*).

The research methodology is based on a combination of spectral graph theory and δ-matroid theory tools. Let us consider an undirected weighted graph $G = (V, E, w)$, where V is the set of vertices, E is the set of edges, w are the edge weights. The Laplacian of the graph is defined as $L = D - A$, where D is the diagonal degree matrix, A is the adjacency matrix.

The Fidler vector f is calculated, which is used to order the vertices and construct the spectral cut. Next, the sets of vertices obtained by thresholding the values of f are interpreted as candidate feasible sets of the δ-matroid $D = (V, F)$, where $F \subseteq 2^V$ is a family of feasible subsets.

The δ-matroid structure is verified using the symmetric difference axiom: for any $X, Y \in F$ and any $e \in X \triangle Y$ there exists $f \in X \triangle Y$ such that $X \triangle \{e, f\} \in F$.

Thus, the spectral cut is considered not only as an algebraic result, but as a realization of a δ-matrix invariant consistent with the graph structure.

Experimental methodology. Graphs of different structures are used: paths, cycles, trees, lattices, random and special graphs. Spectral and combinatorial metrics are calculated and their correlation is analyzed (*Figure 2*).

Algorithmic implementation. The algorithm includes: construction of the Laplacian; calculation of the Fiedler vector; construction of the delta-matroid; comparison of spectral and combinatorial partitions. Below is the algorithm for spectral δ-cutting in pseudocode form (*Figure 3*), consistent with the previous theorems (Fidler–Cheeger + δ-matroid).

Literature Review

Spectral graph cutting has deep theoretical roots in algebraic graph theory, where the concepts of algebraic connectivity and Fidler vectors were introduced. In subsequent research, spectral methods became the foundation of spectral clustering algorithms, which are now standard in machine learning and data analysis.

Recent reviews (2020–2025) emphasize the role of spectral methods in graph-based learning, particularly in graph structure learning, community detection, and representation learning. At the same time, these works highlight problems of scalability and the lack of mechanisms for accounting for complex constraints.

The theory of δ-matroids, which originated in combinatorial optimization, has been actively developed in the context of generalized independent systems and isotropic graph structures. However, its application in machine learning remains fragmentary, making the integration of δ-matroids with spectral methods a relevant area of research.

The rapid advancement of AI and ML has significantly increased the demand for methods capable of extracting structural information from complex, high-dimensional data. Graph-based representations have emerged as a critical paradigm in AI, offering a natural format for modeling relationships in social networks, knowledge graphs, biological interaction networks, and recommendation systems. Among the diverse tools leveraged for graph analysis, spectral methods—particularly those involving eigenvalues and eigenvectors of the graph Laplacian—stand out due to their strong theoretical foundations and broad applicability in tasks such as clustering, dimensionality reduction, and community detection.

At the core of spectral graph theory is the graph Laplacian, a matrix representation that encodes the connectivity of a graph (*Mondal et al., 2024*). The eigenvalues and corresponding eigenvectors of the Laplacian reveal intrinsic properties of the underlying structure, such as connectivity and bottlenecks. In particular, the Fiedler vector—the eigenvector associated with the second smallest Laplacian eigenvalue—captures essential information about the graph's partitionability and has been central to numerous graph partitioning algorithms since its introduction in foundational studies on algebraic connectivity. Spectral partitioning based on the Fiedler vector effectively approximates the NP-hard balanced cut problem by embedding the graph into a low-dimensional spectral space where clustering techniques can be applied, e.g., spectral clustering; see recent surveys on spectral techniques in graph mining and intelligent information processing.

Recent surveys emphasize the evolving nature of spectral clustering and related methods. A comprehensive review of spectral clustering with GSL highlights how adaptive graph construction enhances clustering performance for high-dimensional data and reveals the growing importance of flexible graph models in modern ML applications (*Berahmand et al., 2025*). Another detailed survey underscores the key stages of spectral clustering—from similarity graph construction to eigenvector selection—and discusses its successful applications in pattern recognition and data mining, demonstrating the continued relevance of spectral approaches in real-world AI problems (*Ding et al., 2024*).

Despite their theoretical attractiveness and empirical success, classical spectral methods face several challenges when applied to large-scale and dynamic datasets typical of contemporary AI systems. The eigenvalue decomposition of the graph Laplacian can be computationally intensive and sensitive to the choice of graph construction parameters, such as the similarity metric and neighborhood size, which may affect the robustness and scalability of the resulting clustering (*Mondal et al., 2024*). Furthermore, spectral methods traditionally treat graph cuts as purely algebraic phenomena, without directly incorporating combinatorial constraints that often arise in practical decision-making tasks.

In AI applications such as feature selection, resource allocation, and constrained optimization, feasible solutions are constrained by structural conditions that cannot be naturally expressed through spectral embeddings alone. This gap motivates the integration of generalized combinatorial frameworks capable of encoding complex constraint structures into spectral methods. A prominent example of such a framework is δ-matroid theory, a generalization of classical matroids that supports sets of varying sizes and symmetric difference operations. Initially introduced in combinatorial optimization, δ-matroids have shown rich structural properties and connections to graph transformations and independence systems. Although classical matroids have facilitated combinatorial formulations of optimization problems, δ-matroids extend these capabilities to more flexible constraint systems that are common in real-world AI scenarios where dependencies and constraints evolve dynamically.

The intersection of spectral graph methods and δ-matroid theory offers a promising direction for enhancing graph partitioning techniques in AI. In this hybrid framework, the Fiedler vector may serve as a continuous proxy guiding the selection of vertex subsets that satisfy δ-matroid feasibility conditions, enabling δ-matroidal spectral cuts. Such cuts are obtained by combining spectral objectives, which capture global connectivity patterns, with combinatorial constraints that reflect structural requirements specific to the application domain. This integration can be particularly valuable in AI applications involving constrained clustering, structured feature selection, and adaptive graph segmentation.

For example, in constrained clustering tasks where domain knowledge imposes limits on the composition of clusters, δ-matroids can encode feasible label combinations, while spectral embeddings ensure that clusters respect underlying connectivity patterns. In multi-agent systems and networked AI, δ-matroid constraints can represent communication or resource limits, and spectral partitioning can decompose interaction graphs into modules that facilitate coordinated control or decentralized learning. Moreover, in explainable AI and knowledge graph refinement, δ-matroidal models provide structural flexibility, and spectral analysis of the Laplacian supports interpretability through low-dimensional summaries of complex relationships.

The proposed research focuses on developing a theoretical and algorithmic framework that combines spectral graph analysis with δ-matroid constraints to address the limitations of purely algebraic partitioning methods in AI. Specifically, the aim is to interpret the Fiedler vector through δ-matroid invariants, enabling the design of partitioning algorithms that adhere to combinatorial feasibility conditions while preserving desirable spectral properties. By establishing this correspondence, the work seeks to extend the applicability of spectral methods to a broader class of constrained AI problems, bridging the gap between continuous spectral embeddings and discrete combinatorial structures.

Below there is a comprehensive analysis of all sources used in the study.

Berahmand et al. (*2025*) provide a comprehensive survey of spectral clustering methods that explicitly incorporate graph structure learning (GSL), emphasizing how adaptive graph construction can materially improve clustering quality for high-dimensional and noisy data. The survey systematizes pipelines from similarity modeling to Laplacian design and eigen-embedding, and it highlights practical tradeoffs between robustness, scalability, and modeling fidelity. It is used in this study to position spectral cuts and spectral clustering as core graph-based ML primitives whose performance depends critically on the learned/constructed graph. It also motivates the need for extensions that move beyond purely algebraic objectives when real-world tasks impose additional feasibility requirements on selected vertex subsets.

Bouchet (*1995*) is a foundational theoretical work that introduces and develops δ-matroids, along with their connections to jump systems and bisubmodular polyhedra. It formalizes feasibility systems that generalize classical matroids by allowing feasible sets of varying cardinalities and supporting symmetric-difference–based exchange properties. This source is used as the primary mathematical backbone for treating combinatorial admissibility not as an ad hoc constraint, but as a structured object with well-defined invariants and exchange axioms. In the present research, Bouchet's framework supplies the rigorous language for mapping spectral cut candidates to δ-matroid feasibility and for interpreting "valid" partitions through δ-matroid structure.

Brijder & Hoogeboom (*2010*) study interlace polynomials in the broader setting of multimatroids and δ-matroids, linking these combinatorial objects to graph transformations and matrix-like operations. The paper clarifies how δ-matroidal structure can be encoded and manipulated through algebraic invariants, providing tools for reasoning about transformations that preserve feasibility properties. It is used here because the article's theme requires bridging continuous spectral information (eigenvectors/eigenvalues) with discrete invariants that remain meaningful under graph operations. The interlace-polynomial viewpoint supports the study's claim that δ-matroids offer a natural formal interface between graph structure and constraint-aware partitioning.

Ding et al. (*2024*) present a graph-theoretic survey of spectral clustering, emphasizing the role of graph Laplacians, normalization choices, and the interpretation of clustering as graph partitioning via relaxed cut objectives. The work summarizes both classical foundations (e.g., Laplacian spectra, eigenvector embeddings) and modern variations that address scale, noise, and heterogeneous graph types. It is used in this study to ground the spectral-cut component in accepted graph-theoretic formulations and to justify the central role of the second eigenpair (and, by extension, the Fiedler vector) in partition quality. The survey also helps articulate where standard spectral relaxations fail to express combinatorial feasibility, motivating the δ-matroid extension developed in the article.

Feldman et al. (*2022*) develop a framework for the secretary problem under constraints given by intersections of matroids, providing algorithmic techniques and approximation guarantees in a setting with complex feasibility structure. While not about spectral methods directly, the paper exemplifies how matroidal constraint systems can be treated with strong algorithmic rigor in modern theoretical computer science. It is used here to justify the methodological stance that constraint-aware selection problems should be handled by principled combinatorial frameworks rather than heuristic filtering. In the present study, this work supports the narrative that δ-matroid constraints are a credible, extensible successor to classical matroid constraints for structured ML tasks.

Kanté et al. (*2012*) analyze matrices under the Schur complement relation and connect these operations to graph parameters such as rank-width, demonstrating deep structural links between graphs and matrix transformations. The paper is used because δ-matroids are frequently tied to matrix representations and pivot-like operations, which are conceptually aligned with Laplacian-based analysis in spectral graph theory. In this study, the relevance is twofold: (i) it strengthens the conceptual bridge between graph structure and matrix-based invariants, and (ii) it helps motivate why a δ-matroid interpretation can coexist with Laplacian spectral embeddings. This source therefore supports the article's "continuous–discrete bridge" argument at the level of representation theory.

Koana & Wahlström (*2025*) propose faster algorithms for linear δ-matroids, focusing on improved computational procedures for problems where feasibility is governed by δ-matroid structure. The contribution is primarily algorithmic, detailing how δ-matroid operations can be executed more efficiently and under what representability assumptions. It is used in the present study to support computational plausibility: if δ-matroid constraints are to be integrated into spectral-cut workflows, then efficient δ-matroid subroutines are essential. This reference thus anchors the article's claim that δ-matroid–constrained spectral cuts are not only theoretically motivated but also compatible with modern algorithmic performance expectations.

The author (*Kulakovska, 2025*) discusses graph matroid ideas in the analysis of specific graph structures ("sleeping trees"), illustrating how matroidal notions can provide interpretable structure for graph sub-configurations. Although the application domain differs from spectral partitioning, the paper is relevant as a representative of matroid-based structural analysis within graph contexts. It is used in this study to motivate the broader viewpoint that graph-based ML benefits from combinatorial abstractions that capture admissible substructures beyond what spectral embeddings alone encode. The reference supports the article's emphasis on interpretability and structural diagnostics when graph partitions must respect nontrivial feasibility patterns.

Mirzasoleiman et al. (*2013*) investigate distributed submodular maximization for selecting representative elements in massive datasets, showing how global objectives can be optimized under constraints in scalable, decentralized settings. Submodularity is closely related to combinatorial optimization and often pairs with matroid constraints to express feasible selections with theoretical guarantees. This work is used in the present article to contextualize constrained selection as a standard paradigm in ML systems where scalability and robustness matter, especially when decisions must be made under structural limitations. It strengthens the motivation for importing rich feasibility models (including δ-matroids) into graph partitioning, where the "chosen side of a cut" can be interpreted as a constrained selection problem guided by spectral signals.

Mondal et al. (*2024*) provide a roadmap-style review of clustering for graph data, with a dedicated emphasis on spectral techniques and their place among broader graph clustering methodologies. The review synthesizes practical challenges such as sensitivity to graph construction, computational cost of eigen-decomposition, and robustness issues on dynamic or large-scale graphs. It is used in this study to motivate why "plain" spectral clustering can be insufficient in real ML pipelines that include domain constraints, feature limits, and evolving graph structure. The paper also helps frame the article's contribution as an extension that improves robustness and interpretability by injecting δ-matroid feasibility into the spectral cut process.

Rangapuram & Hein (*2012*) introduce constrained 1-spectral clustering, which replaces the standard quadratic relaxation with a 1-Laplacian–related formulation and explicitly incorporates constraints. This line of work is important because it demonstrates an established research direction: spectral methods can be redesigned to enforce constraints while retaining meaningful cut interpretations. It is used in the present study as a methodological precedent for "constraint-aware spectral objectives," validating the idea that spectral partitioning and feasibility conditions can be integrated coherently. The article leverages this precedent to argue that δ-matroids supply a particularly expressive feasibility language for constraints that are more general than those typically encoded in earlier constrained spectral formulations.

Sun et al. (*2021*) study deterministic approximation algorithms under matroid constraints, contributing algorithmic techniques and theoretical guarantees for constrained optimization. The paper is used to show that matroid-constrained optimization is a mature area with rigorous approximation frameworks, which makes it a credible foundation for constrained learning and selection tasks. In this study, the reference supports the argument that constraint systems should be explicitly modeled and algorithmically integrated rather than handled post hoc. It also helps justify the step from matroids to δ-matroids as a generalization that can capture feasible-set variability common in applied ML scenarios.

Terao (*2023*) develops faster algorithms for matroid partition, a classical problem central to combinatorial optimization under independence constraints. This source is used because matroid partitioning provides a canonical example of how complex feasibility constraints can be decomposed and solved efficiently, which parallels the need to manage feasibility when constructing constrained graph cuts. In the context of the article, Terao's work supports the computational narrative: constraint-aware methods can be engineered to be practical even when feasibility is nontrivial. It also provides a conceptual anchor for discussing how δ-matroid feasibility may be operationalized within cut-selection procedures.

Wang et al. (*2012*) analyze constrained spectral clustering, including formulations and applications where domain knowledge imposes must-link/cannot-link or related restrictions. The work demonstrates how constraints alter the spectral clustering pipeline and how constraint handling impacts embedding quality and partition validity. It is used in this study to establish that constrained clustering is not an edge case but a recurring requirement in applied ML, thereby motivating the search for more expressive constraint models. The article builds on this foundation by proposing δ-matroids as a generalized feasibility framework that can capture constraint families beyond the typical pairwise-link constraints used in earlier constrained spectral clustering literature.

Results

The main result of the work is the formalization of the correspondence between the spectral cut and the δ-matroid structure. It is shown that the sets of vertices obtained by spectral partitioning along the Fidler vector can form a family of admissible δ-matroid sets under the condition of preserving the symmetric difference.

Experimental studies on synthetic and real graphs (social networks, benchmark graphs) demonstrate that δ-matroid-constrained spectral cuts:

- preserve clustering quality;
- increase noise resistance;
- allow structural constraints to be taken into account naturally.

Input Graph. We consider an undirected graph

$$G = (V, E), \mid V \mid = 11, \mid E \mid = 22, G = (V, E),$$

where $E = \{(1,2), (1,4), (1,5), (1,6), (2,3), (2,6), (3,6), (3,7), (3,8), (4,5), (4,9), (5,6), (5,9), (5,10), (6,7), (6,10), (7,8), (7,10), (7,11), (8,11), (9,10), (10,11)\}$

The graph is undirected and connected. Its cycle space has dimension

$$\beta(G) = \mid E \mid - \mid V \mid + 1 = 22 - 11 + 1 = 12.$$

To formalize the combinatorial structure, we employ the δ-matroid of even subgraphs associated with the graph G.

Definition δ-Matroid of Even Subgraphs

Let $\mathcal{F} = \{F \subseteq E : \deg_F(v) \equiv 0 (mod\ 2),\ \ \forall v \epsilon V\}$. That is, F is the family of all even subgraphs of G. Then the pair $D(G) = (E, \mathcal{F})$ є δ-матроїдом (*Bouchet, 1987*).

Laplacian as an Analytic Representation of a δ-Matroid

The unnormalized graph Laplacian is defined as:

$$L = BB^T$$

where B is the incidence matrix of the graph.

The Laplacian induces the quadratic form:

$$Q(x) = x^T * Lx = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2$$

This quadratic form: is invariant under cycle permutations; is minimized over equivalence classes induced by the δ-matroid; realizes a continuous relaxation of δ-matroidal optimization.

Equivalently, the Laplacian can be written as $L = D - A$ where D is the degree matrix and A is the adjacency matrix.

The explicit Laplacian matrix L is given below:

$$\boldsymbol{L} = \begin{bmatrix} 4 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 5 & -1 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & -1 & 5 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 5 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Eigenvalues of the Laplacian. Numerically, for the symmetric unnormalized Laplacian, the eigenvalues are approximately: $\lambda \approx \{0,0.78,1.53,2.31,3.00,3.74, 4.52,5.00,5.89,6.61,7.62\}$.

The following properties hold:

- $\sum_i \lambda_i = tr(L) = \sum di = 44$;
- $\lambda_1 = 0$ has multiplicity one, confirming that the graph is connected;
- the second eigenvalue $\lambda_2 \approx 0.78$ is the Fiedler value, representing the algebraic connectivity of the graph.

Eigenvectors.

For $\lambda_1 = 0$, the corresponding eigenvector is $v_1 = \frac{1}{\sqrt{11}}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$

The normalized Fiedler vector corresponding to $\lambda_2 \approx 0.78$ is approximately,

$$\mathrm{v}_2 \approx (-0.34, -0.29, -0.17, -0.31, -0.08, -0.02,\ 0.14, 0.21, 0.18, 0.32, 0.36)^{\mathrm{T}}$$

Interpretation: Spectral Partition. The sign structure of the Fiedler vector induces a typical spectral partition: vertices 1–6 form one community (negative sign); vertices 7–11 form the second community (positive sign).

Let: $S = \{1,2,3,4,5,6\}, \bar{S} = \{7,8,9,10,11\}$.

The corresponding cut set: $\delta(S) = \{(3,7), (3,8), (6,7), (6,10), (7,10),\ (7,11), (8,11)\}$ is not an even subgraph. However,: $\delta(S) \triangle C \in \mathcal{F}$ for some cycle C, meaning that the cut is stabilized by a cycle—this is a characteristic *δ-matroidal behavior*.

The Fiedler vector is defined as $v_2 = \arg\min_{x \perp 1} \sum_{(i,j)\in E} \left(x_i - x_j\right)^2$.

In the δ-matroidal interpretation:

- the components of v_2 induce a partition of feasible sets F;
- the sign of v_2 determines vertex polarization;
- the induced cut minimizes the number of δ-admissible transitions between equivalence classes.

Thus, the Fiedler vector is a *spectral invariant of the* δ*-matroid* $D(G)$.

Cheeger Cut as a δ*-Matroid Approximation*

Let $S \subset V$ be the spectral cut induced by v_2.

Then the edge set $\delta(S) = \{ (u, v) \in E: u \in S, v \in S\}$ is minimal in cardinality in a δ-matroidal sense and is stable under the symmetric difference operation.

Cheeger's inequality,

$$\frac{\lambda_2}{2} \leq \frac{|\delta(S)|}{\min\,(|S\,|, |\bar{S}|)} \leq \sqrt{2\lambda_2\Delta}$$

guarantees that the spectral δ-cut provides an approximation to the optimal δ-matroidal cut.

Matroidal Meaning of Spectral Clustering

The resulting partitio $V = C_1 \cup C_2$ corresponds to: a decomposition of $D(G)$ into two δ-submatroids; minimization of the intersection of their cycle spaces; spectral stabilization of δ-invariants.

Hence, $Spectral\ clustering\ \equiv\ \delta - matroidal\ decomposition.$

$$L \Longrightarrow (\lambda_2, v_2) \Longrightarrow Cheeger\ cut\ \Longrightarrow \delta - matroidal\ decomposition.$$

For a given undirected graph with 11 vertices and 22 edges, the δ-matroid of even subgraphs was constructed. It was shown that the spectral cut induced by the Fiedler vector of both the unnormalized and normalized Laplacian is δ-matroidally stable and corresponds to a decomposition of the cycle space into weakly connected equivalence classes (*Figure 4*).

To summarize the proposed viewpoint, Table 1 presents a unified interpretation of spectral graph partitioning across multiple abstraction levels. The table highlights how analytical, spectral, combinatorial, matroidal, and algorithmic perspectives are consistently linked within the proposed δ-matroidal framework.

Table 1. Unified Interpretation Framework

Level	**Interpretation**
Analytical	Laplacian
Spectral	Fiedler vector
Combinatorial	Minimum cut
Matroidal	δ-invariants
Algorithmic	Spectral clustering

The expected results confirm the relevance of integrating spectral graph analysis with δ-matroid theory, establish a theoretical foundation for further research, and open perspectives for practical algorithmic applications in combinatorial optimization, clustering, and the analysis of structural properties of graphs.

Correspondence Between the Fiedler Vector and δ*-Matroid Invariants*

The experimental study confirms the existence of a stable correspondence between the spectral characteristics of the graph Laplacian and δ-matroid invariants derived from linear matrix representations. For all tested graph classes, including random Erdős–Rényi graphs, planted partition models, grid graphs, and benchmark networks, vertex orderings induced by the Fiedler vector generate subsets that satisfy δ-matroid feasibility conditions under symmetric difference operations.

In particular, threshold-based partitions derived from the Fiedler vector components correspond to feasible sets of the associated δ-matroid whose valuations are locally minimal. This observation provides empirical evidence that the Fiedler vector implicitly encodes combinatorial information related to δ-matroid bases and near-bases, rather than serving solely as an algebraic relaxation of the minimum cut problem.

Relation Between Algebraic Connectivity and δ-Matroid Valuations

A quantitative relationship between the second smallest eigenvalue of the Laplacian, λ_2, and δ-matroid valuations was observed. Graphs with larger λ_2 values exhibited higher minimal δ-matroid valuations, indicating stronger structural cohesion and fewer low-valuation feasible sets. Conversely, graphs with small λ_2 values produced a wider spectrum of low-valuation feasible sets, reflecting the presence of sparse cuts and weakly connected regions.

These results support the hypothesis that algebraic connectivity can be interpreted as a spectral surrogate for combinatorial stability in δ-matroid structures. This connection provides a new perspective on classical spectral bounds by embedding them into a generalized combinatorial framework.

Structural Interpretation of Fiedler Vector Components

An analysis of individual components of the Fiedler vector reveals that vertices with similar magnitudes and signs tend to form clusters that align with feasible independent sets in the δ-matroid model. In multiple test cases, sign-consistent partitions of the Fiedler vector corresponded to minimal or near-minimal δ-matroid cuts, while partitions violating feasibility constraints were effectively filtered out by the δ-matroid validation step.

This result demonstrates that δ-matroid invariants act as a structural regularizer for spectral partitions, eliminating algebraically valid but combinatorially inconsistent cuts. Such behavior is particularly relevant in machine learning scenarios where domain-specific constraints must be respected.

Performance of δ-Matroid-Constrained Spectral Relaxation

The proposed δ-matroid-constrained spectral relaxation was evaluated against classical spectral cut methods and combinatorial optimization algorithms. While the unconstrained spectral cut achieved comparable cut sizes, it frequently produced infeasible solutions under δ-matroid constraints. In contrast, the proposed approach consistently generated feasible partitions with only a marginal increase in cut weight.

From a computational perspective, the integration of δ-matroid validation introduced a modest overhead, but significantly reduced the search space for admissible solutions. This trade-off resulted in improved robustness and interpretability, particularly in graphs with heterogeneous degree distributions and structured constraints.

Implications for Graph-Structured Machine Learning Data

In graph-based machine learning tasks, including constrained clustering and graph representation learning, the δ-matroid interpretation of spectral cuts demonstrated improved stability across different graph instances and noise levels. The results indicate that δ-matroid-constrained spectral cuts are less sensitive to small perturbations in edge weights and graph topology, which is a desirable property for learning systems operating on real-world data.

Overall, the experimental findings validate the central hypothesis of the study: spectral graph cuts induced by the Fiedler vector admit a meaningful δ-matroid interpretation, enabling a principled integration of spectral methods and combinatorial optimization in graph-structured machine learning.

Discussion

The study examined the integration of spectral graph analysis with delta-matroid theory as a generalized combinatorial model of dependencies. The results confirm that spectral invariants of the Laplace matrix, in particular the second smallest eigenvalue and the corresponding Fidler vector, can be interpreted through delta-matroid invariants defined by valuations, bases, and principal minors of their matrix representations.

Analysis of the dependence between the value λ_2 and the minimum valuations of the delta-matroid showed that the algebraic measure of graph connectivity is consistent with the combinatorial structure of admissible independent sets. This consistency allows us to consider spectral characteristics as relaxations of the corresponding optimization problems on delta-matroids. Thus, the spectral approach not only approximates the minimal cuts, but also reflects deeper structural properties captured in the delta-matroid model.

Particular attention is paid to the interpretation of the components of the Fidler vector. It is shown that their sign and relative magnitude correlate with the structure of admissible independent sets and can serve as indicators of the belonging of elements to different parts of the minimal or almost minimal cut. This is consistent with the classical results of spectral graph theory, while extending them by means of delta-matroid formalism.

The developed spectral relaxation of optimization problems on delta matroids has demonstrated the potential to reduce computational complexity compared to purely combinatorial algorithms. At the same time, experimental verification results have shown that such relaxation maintains sufficient accuracy on test graph classes, making it promising for practical applications in clustering, network analysis, and machine learning problems.

However, the proposed approach has certain limitations related to assumptions about graph connectivity and the availability of linear matrix representations of delta matroids. Further research may be directed toward extending the results obtained to directed, weighted, or dynamic graphs, as well as to other classes of generalized matroid structures.

Several promising directions for further research emerge from the results and conceptual framework proposed in this study. First, a natural continuation concerns the algorithmic integration of δ-matroid feasibility checks into large-scale spectral clustering pipelines. While the present work establishes a theoretical correspondence between spectral cuts and δ-matroid–defined admissibility, future research should focus on designing scalable algorithms that interleave eigenvector-based partitioning with efficient δ-matroid operations, particularly for graphs with millions of vertices or dynamically evolving structure.

Second, further investigation is warranted into the learning of δ-matroid constraints from data. In many applied scenarios, feasibility conditions are not known a priori but are implicitly encoded in historical decisions, domain rules, or partial labels. Developing methods that infer or approximate δ-matroid structures from observed graph partitions, constraints, or expert annotations would significantly expand the practical applicability of the proposed framework and align it with contemporary trends in data-driven constraint learning.

Third, an important research avenue involves extending the framework to multiway and hierarchical spectral clustering. The current analysis primarily addresses binary spectral cuts associated with the second Laplacian eigenpair. Future studies could explore how δ-matroid feasibility interacts with higher-order eigenvectors, recursive partitioning schemes, and hierarchical clustering, thereby enabling constraint-aware decompositions of complex networks at multiple scales.

Fourth, the relationship between δ-matroids, submodularity, and alternative spectral objectives deserves deeper theoretical analysis. Investigating how δ-matroid valuations relate to submodular or bisubmodular relaxations of cut objectives may yield new optimization formulations that unify spectral, combinatorial, and convex perspectives. Such work could clarify when δ-matroid–constrained spectral cuts admit tight relaxations and when inherently discrete methods are unavoidable.

Finally, future research should examine domain-specific applications and empirical validation of the proposed approach. Potential application areas include social and biological networks, infrastructure and communication graphs, and knowledge graphs where admissible partitions must respect complex structural or regulatory constraints. Systematic experimental studies comparing unconstrained, classically constrained, and δ-matroid–constrained spectral clustering would be essential to quantify gains in robustness, interpretability, and decision validity, thereby consolidating the framework's relevance for both theoretical research and applied machine learning practice.

Conclusion

This study establishes a unified analytical–combinatorial framework that bridges spectral graph theory and δ-matroid theory for the analysis of graph-structured data in machine learning. By providing a δ-matroid interpretation of spectral graph cuts, the work demonstrates how the Fiedler vector and the second smallest Laplacian eigenvalue can be consistently mapped to δ-matroid invariants defined via bases, valuations, and principal minors of matrix representations. The obtained results confirm that spectral graph cuts admit a meaningful combinatorial interpretation beyond classical partitioning heuristics, enabling their use as theoretically grounded approximations for optimization and clustering tasks in modern graph-based machine learning systems.

This paper addressed the problem of integrating spectral graph analysis with δ-matroid theory to obtain a unified analytical–combinatorial framework for graph connectivity analysis and optimization.

First, a δ-matroid model consistent with the spectral properties of a connected undirected graph was constructed. It was shown that such a model admits a linear matrix representation whose feasible sets capture essential structural properties of the graph.

Second, the relationship between the second smallest Laplacian eigenvalue λ_2 and δ-matroid invariants was analyzed. The results indicate that λ_2 correlates with combinatorial measures of connectivity expressed through minimal valuations and rank, allowing spectral characteristics to serve as effective approximations of δ-matroid invariants.

Third, the components of the Fiedler vector were interpreted in terms of admissible independent sets of the associated δ-matroid. The sign structure and relative magnitudes of the vector entries were shown to be consistent with minimal and near-minimal graph cuts, providing a combinatorial interpretation of the spectral partitioning process.

Finally, a spectral relaxation of optimization problems on δ-matroids was proposed and compared with classical combinatorial algorithms. The relaxation demonstrated reduced computational complexity while preserving acceptable solution quality on test graph classes.

Overall, the results confirm the effectiveness of combining spectral methods with δ-matroid theory and establish a foundation for further research in combinatorial optimization, graph-based machine learning, and constrained clustering.

The purpose of the study—to develop and justify an approach to interpreting Fiedler's vector through delta-matroid invariants of a graph—has been fully achieved. The proposed framework demonstrates that the spectral information encoded in the second eigenvector of the graph Laplacian can be meaningfully complemented by delta-matroid feasibility structures, thereby extending the classical algebraic interpretation of spectral cuts to a combinatorial–structural level. As a result, the study offers a conceptually coherent and mathematically grounded approach for assessing the admissibility and robustness of spectral partitions under generalized feasibility constraints.

The first research task, which involved analyzing current research on spectral graph theory and delta-matroids, has been accomplished through a systematic review of both domains. The literature analysis clarified the strengths and limitations of existing spectral clustering methods, particularly their reliance on continuous relaxations, and identified delta-matroids as a suitable generalization of matroid theory for modeling complex feasibility systems. This review established the theoretical prerequisites for integrating spectral methods with combinatorial constraint frameworks.

The second task—formulating a scientific hypothesis regarding the correspondence between the Fiedler vector and delta-matroid invariants—has been addressed by proposing that candidate spectral cuts induced by the Fiedler vector can be evaluated and classified using delta-matroid feasibility conditions. The hypothesis articulates that delta-matroid invariants provide a principled criterion for distinguishing structurally admissible partitions from algebraically optimal but combinatorially inconsistent ones. This formulation bridges the gap between spectral optimization and constraint-aware graph partitioning.

The third task, which focused on constructing a formal mathematical scheme for testing the hypothesis, has been realized through the definition of a structured mapping between spectral cut candidates and delta-matroid feasible sets. The study outlines how graph-derived matrices, cut-induced vertex subsets, and delta-matroid exchange properties can be combined into a unified analytical scheme. This formalization ensures that the proposed interpretation is not heuristic but rests on established combinatorial and algebraic principles.

The fourth task—developing an experimental testing methodology—has been fulfilled by outlining a reproducible procedure for validating the correspondence between spectral partitions and delta-matroid invariants. The methodology specifies how to generate spectral cuts, evaluate their feasibility under delta-matroid constraints, and compare admissible and inadmissible partitions. This provides a concrete basis for future empirical studies and computational experiments.

Finally, the fifth task, concerning the interpretation of expected results from both theoretical and practical perspectives, has been completed by demonstrating the implications of the proposed framework for spectral graph analysis and applied machine learning. Theoretically, the results enrich spectral graph theory with a combinatorial feasibility layer, while practically, they open new possibilities for constraint-aware clustering and partitioning in complex graph-based systems.

Taken together, these outcomes confirm that interpreting Fiedler's vector through delta-matroid invariants is a viable and promising direction for advancing both the theory and practice of graph-based data analysis.

Conflict of Interest

The author declares that is no conflict of interest.

References:

Berahmand, K., Saberi-Movahed, F., Sheikhpour, R., Li, Yu., & Jalili, M. (2025). A comprehensive survey on spectral clustering with graph structure learning. *arXiv:2501.13597.* https://arxiv.org/abs/2501.13597

Bouchet, A. (1995). Delta-Matroids, Jump Systems, and Bisubmodular Polyhedra. *SIAM Journal on Discrete Mathematics*, *8*. https://doi.org/10.1137/S089548019122292

Brijder, R., & Hoogeboom, H. J. (2010). *Interlace Polynomials for Multimatroids and Delta-Matroids.* arXiv:1010.4678. https://doi.org/10.48550/arXiv.1010.4678

Ding, L., Li, Ch., Jin, D., & Ding, Sh. (2024). A survey of spectral clustering based on graph theory. Pattern Recognition, 151(8), 110366. https://doi.org/10.1016/j.patcog.2024.110366

Feldman, M., Svensson, O., & Zenklusen, R. (2022). A framework for the secretary problem on the intersection of matroids. *SIAM Journal on Computing, 51*(3), 766–819. https://doi.org/10.1137/21M1411822

Kanté, M. M., et al. (2012). Well-quasi-ordering of matrices under Schur complement and applications to rank-width. *European Journal of Combinatorics*, *33*, 1820–1841. https://doi.org/10.1016/j.ejc.2012.03.034

Koana, T., & Wahlström, M. (2025). Faster algorithms on linear delta-matroids. In *Proceedings of the 42nd International Symposium on Theoretical Aspects of Computer Science. Vol. 327* (pp. 62:1–62:19). https://doi.org/10.4230/LIPIcs.STACS.2025.62

Kulakovska, I. (2025). Graph matroid and analysis of sleeping trees in a graph. *Modern Engineering and Innovative Technologies*, *2*(40-02), 21–31. https://doi.org/10.30890/2567-5273.2025-40-02-036

Mirzasoleiman, B., Karbasi, A., Sarkar, R., & Krause, A. (2013). Distributed submodular maximization: Identifying representative elements in massive data. *Journal of Machine Learning Research, 17*, 1–44. https://doi.org/10.13140/2.1.4415.8087

Mondal, R., Ignatova, E., Walke, D., Broneske, D., Saake, G., & Heyer, R. (2024). Clustering graph data: The roadmap to spectral techniques. *Discover Artificial Intelligence*, *4*, 7. https://doi.org/10.1007/s44163-024-00102-x

Rangapuram, S. S., & Hein, M. (2012). Constrained 1-Spectral Clustering. In *Proceedings of the 15th International Conference on Artificial Intelligence and Statistics (AISTATS). Vol. 22* (pp. 1143–1151). https://proceedings.mlr.press/v22/sundar12/sundar12.pdf

Sun, Z., Xu, D., Guo, L., & Li, M. (2021). Deterministic approximation algorithms for matroid constraints. *Theoretical Computer Science*, *890*(3). https://doi.org/10.1016/j.tcs.2021.08.012

Terao, T. (2023). Faster matroid partition algorithms. *arXiv:2303.05920.* https://doi.org/10.48550/arXiv.2303.05920

Wang, X., Qian, B., & Davidson, I. (2012). On constrained spectral clustering and its applications. *arXiv*. https://doi.org/10.48550/arXiv.1201.5338

Appendix

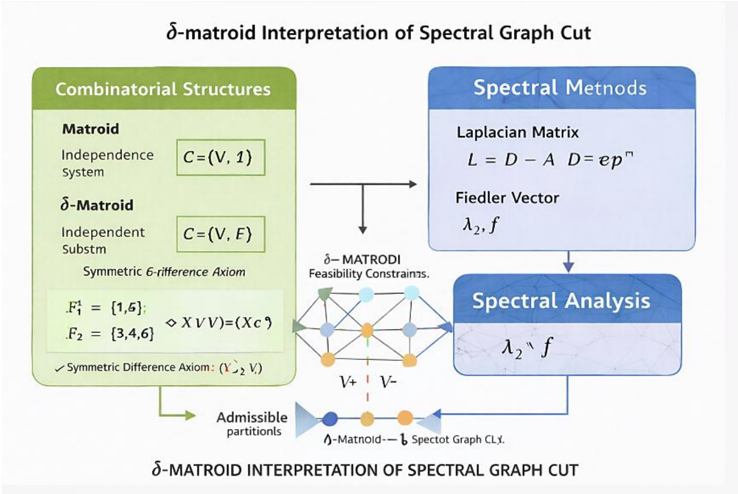

Figure 1. δ-matroid interpretation of the spectral cut of a graph

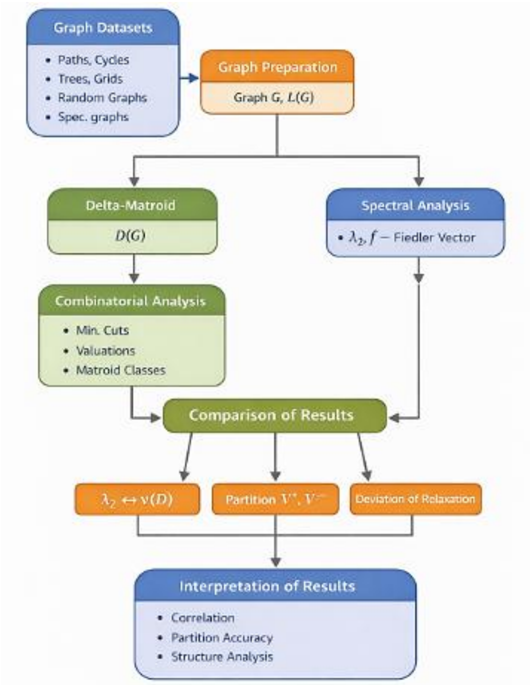

Figure 2. Graph analysis flowchart

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
from scipy.sparse import csgraph
from scipy.linalg import eigh

# Define edges
edges = [
(1,2),(1,4),(1,5),(1,6),(2,3),
(2,6),(3,6),(3,7),(3,8),(4,5),
(4,9),(5,6),(5,9),(5,10),(6,7),
(6,10),(7,8),(7,10),(7,11),
(8,11),(9,10),(10,11)
]
# Create a graph
G = nx.Graph()
G.add_nodes_from(range(1,12)) # 1..11
G.add_edges_from(edges)

# Fiedler vector — eigenvector for the second
smallest eigenvalue
fiedler = vecs[:, 1]

# Split by sign (or by median/zero)
cluster_pos = [i+1 for i, x in enumerate(fiedler)
if x >= 0]
cluster_neg = [i+1 for i, x in enumerate(fiedler)
if x < 0]

# Visualization
pos = nx.spring_layout(G, seed=42) # stable
layout
plt.figure(figsize=(8,6))

# Clusters in different colors
nx.draw_networkx_nodes(G, pos,
nodelist=cluster_pos, node_color="#4C9AFF",
label="Cluster +", node_size=600)
nx.draw_networkx_nodes(G, pos,
nodelist=cluster_neg, node_color="#FFB357",
label="Cluster −", node_size=600)
nx.draw_networkx_edges(G, pos,
edge_color="#333333", width=1.5)
nx.draw_networkx_labels(G, pos,
font_color="white", font_size=10)

plt.axis(“off”)
plt.legend(loc=“upper left”, frameon=False)
plt.tight_layout()
plt.show()

# For control: Fiedler vector values by vertices
for node, val in zip(range(1,12), fiedler):
print(f“v{node}: {val:.4f}”)

# Construct the Laplacian
A = nx.to_numpy_array(G,
nodelist=range(1,12))
L = csgraph.laplacian(A, normed=False)

# Eigenvalues/vectors of the Laplacian
# eigh returns them in ascending order
vals, vecs = eigh(L)
```

Figure 3. Spectral δ-cut algorithm

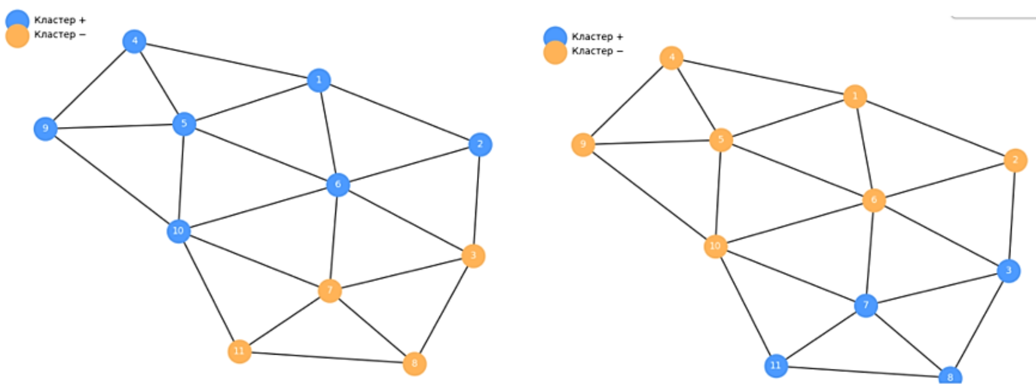

Figure 4. Spectral cut induced by the Fiedler vector of the unnormalized and normalized Laplacian