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Oleg V. Lazorenko, Doctor of Science in Physics and Mathematics, Assistant Professor, Head of General Physics Department, School of Physics, V. N. Karazin Kharkiv National University. Kharkiv, Ukraine.
E-mail: Oleg.V.Lazorenko@karazin.ua
ORCID 0000-0002-0250-8671, Scopus 8221324700
Andriy A. Onishchenko, Candidate of Science in Physics and Mathematics, Assistant Professor of Physics Department, Faculty of Automation, Computer-Integrated Technologies and Systems, Kharkiv National University of Radioelectronics. Kharkiv, Ukraine.
E-mail: andrey.onishchenko@nure.ua
ORCID 0000-0002-2118-9119, Scopus 56825402600
Maryna A. Udovenko, Candidate of Science in Pedagogics, Researcher of Low-Temperature Physics Department, School of Physics, V. N. Karazin Kharkiv National University. Kharkiv, Ukraine.
E-mail: marinaudovenko877@gmail.com
ORCID 0009-0006-9969-703X

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Improved Hurst Fractal Dimension Estimation with the Corrective Function Method Application

Abstract: The relevance of this study is determined by the increasing importance of accurate mono-fractal and multi-fractal analysis methods for investigating complex natural and artificial processes. The Hurst fractal dimension is one of the key numerical characteristics used to describe self-affine properties, persistence, anti-persistence and memory effects in physical signals. However, when the rescaled range analysis method, or R/S method, is applied to finite stochastic signals, the accuracy of Hurst fractal dimension estimation may be significantly reduced. The research problem lies in the existence of an undesirable non-linear dependence between the estimated and true values of the Hurst fractal dimension obtained by the R/S method. The scientific novelty of the article consists in constructing and applying a corrective function that compensates for this distortion and improves estimation accuracy. The object of the study is Hurst fractal dimension estimation by means of rescaled range analysis. The aim is to investigate the specific features of this estimation procedure, improve its accuracy using the corrective function method, and test the proposed approach on model stochastic mono-fractal signals and the sunspot number time series. The methodological framework includes the R/S method, the corrective function method, the wavelet transforms modulus maxima method, and multifractal detrended fluctuation analysis. The corrective function was built using stochastic mono-fractal signals based on the modified cosine Weierstrass–Mandelbrot function. The empirical part of the study uses daily total sunspot numbers from 1 January 1818 to 31 July 2024. The theoretical basis includes works on fractal geometry, chaos theory, multifractals, Hurst analysis, wavelet analysis and solar activity by B. Mandelbrot, K. Falconer, J. Feder, H. Hurst, S. Mallat, J. Kantelhardt and co-authors, L. Chernogor, O. Lazorenko, A. Onishchenko and other researchers. The results demonstrate that the R/S method produces systematic non-linear distortions: depending on signal length, Hurst fractal dimension estimates may be overestimated in one range and underestimated in another. The corrective function method reduces this distortion by inverting the empirically obtained dependence between estimated and true fractal dimensions. Its application to the sunspot number time series reveals several scale ranges with mono-fractal properties, confirms the multifractal and non-stationary character of the process, and supports its interpretation as a second-order fractal process. The study concludes that the corrective function method is an effective tool for improving the reliability of Hurst fractal dimension estimation when using the R/S method. The proposed approach may be useful for analysing finite stochastic signals and complex natural processes, including solar activity, geospace phenomena and non-linear dynamical systems.

Keywords: signal processing, mono-fractal analysis, corrective function, rescaled range analysis, Hurst fractal dimension, sunspot numbers time series.

Abbreviations:

CWT is Continuous Wavelet Transform,
FGN is Fractional Gaussian Noise,
GBM is Generalized Brownian Motion,
LRD is Long-Range Dependence,
$MFDFA$ is Multi-Fractal Detrended Fluctuation Analysis,
SDF is Spectral Density Function,
SRD is Short-Range Dependence,
$WTMM$ is Wavelet Transform Module Maxima.

Introduction

The term *fractal*, derived from the Latin *fractus*, meaning "broken", was introduced into scientific circulation by the American physicist and mathematician Benoit Mandelbrot. After several decades of intensive scientific discussion involving both supporters and opponents of the new approach, fractal ideas became firmly established in modern science and acquired the status of an important methodological basis for the analysis of complex natural and artificial processes (*Mandelbrot, 1982*; *Falconer, 1990*; *Feder, 1988*; *Feldman, 2012*; *Mandelbrot, 2005*; *Schroeder, 1991*).

According to the so-called fractal paradigm proposed by V. V. Yanovsky, fractality, like nonlinearity before it, may be regarded as one of the fundamental properties of the world around us (Yanovsky, 2003, 2006). In the nonlinear and system paradigms formulated by L. F. Chernogor, open, complex, nonlinear and dynamical systems are considered as systems in which many natural and artificial processes, especially those caused by powerful non-stationary sources of energy release, may be classified as short-time, ultra-wideband, nonlinear and fractal processes (*Chernogor, 2008*). Thus, fractality is not an abstract mathematical construction only; it is increasingly interpreted as a universal property of physical reality and as an effective tool for describing complex processes in nature and technology.

To discover, investigate, describe and explain the fractal properties of natural and artificial processes, various methods of mono-fractal and multi-fractal analysis are required. These methods provide a set of numerical characteristics that make it possible to reflect the main peculiarities of the signals and processes under investigation (*Lazorenko & Chernogor, 2023*). Such characteristics have become especially important in applied physics, radio physics, geophysics, solar-terrestrial physics, signal processing, engineering and other fields where researchers deal with nonlinear, non-stationary and scale-dependent phenomena.

A fractal dimension is one of the most important numerical characteristics used in fractal analysis. In general, many different fractal dimensions can be estimated for mathematical and physical, or natural, fractal signals and processes. Each of them has its own calculation procedure and its appropriate field of application (*Lazorenko & Chernogor, 2020*; *Lazorenko & Chernogor, 2023*). In this paper, attention is focused on one such characteristic, namely the Hurst fractal dimension (D), and on one of the oldest and most widely used methods for its estimation, the rescaled range analysis method, which is often referred to as the R/S method.

The relevance of the study is determined by the fact that the R/S method remains one of the classical and reliable tools of mono-fractal analysis and is widely applied in different branches of modern science and technology. At the same time, the accuracy of estimating the Hurst fractal dimension by this method is of fundamental importance, since the obtained numerical value directly affects the interpretation of fractal properties, persistence, anti-persistence and memory effects in the analysed process. Therefore, any systematic distortion in the estimation of the Hurst fractal dimension may lead to incorrect conclusions about the physical nature of the signal.

The research problem addressed in this paper is connected with the existence of an undesirable nonlinear dependence between the estimated value of the Hurst fractal dimension and its true value when the R/S method is applied. In an ideal case, the estimated value should coincide with the true one and should not depend on the finite length of the signal. In practice, however, the estimation obtained by a mono-fractal analysis method may be distorted by a nonlinear function depending on both the true fractal dimension and the number of signal points. This circumstance worsens the accuracy of Hurst fractal dimension estimation and becomes a negative factor for mono-fractal analysis as a whole.

The working hypothesis of the study is that the accuracy of Hurst fractal dimension estimation by the R/S method can be improved if the nonlinear dependence between the estimated and true values is determined on the basis of specially generated model stochastic mono-fractal signals and then compensated by means of a corrective function. In this case, the corrective function method may reduce systematic estimation errors and make the R/S method more reliable for the analysis of finite physical signals.

The scientific novelty of the paper lies in the construction and application of a corrective function for improving the estimation accuracy of the Hurst fractal dimension obtained by the R/S method. The corrective function is built on the basis of a specially created set of model stochastic mono-fractal signals with known and controlled values of the fractal dimension and signal length. This makes it possible to analyse the nonlinear dependence between the estimated and true fractal dimension values and to compensate for its influence. In addition, the proposed approach is tested not only on model signals, but also on a real natural physical process represented by the sunspot number time series.

The object of the investigation is the calculation of the Hurst fractal dimension using the R/S method.

The subject of the investigation is the improvement of the accuracy of Hurst fractal dimension estimation when the R/S method is applied to finite stochastic and natural physical signals.

The study aims to investigate the peculiarities of Hurst fractal dimension estimation by the R/S method, to improve the accuracy of this estimation through the corrective function method, and to analyse the corresponding practical results obtained for the sunspot number time series as a well-known fractal natural process.

To achieve this aim, the following research tasks are set:

- to consider the theoretical basis of the R/S method and its role in estimating the Hurst fractal dimension;
- to analyse the relationship between the Hurst exponent and the Hurst fractal dimension within the framework of the generalized Brownian motion model;
- to demonstrate the existence of nonlinear dependence between the estimated and true values of the Hurst fractal dimension when the R/S method is applied;
- to construct a corrective function using model stochastic mono-fractal signals with controlled parameters;
- to assess the effectiveness of the corrective function method for increasing estimation accuracy;
- to apply the R/S method and the corrective function method to the sunspot number time series;
- to compare mono-fractal results with multi-fractal characteristics obtained by the wavelet transform modulus maxima method and multifractal detrended fluctuation analysis;
- to evaluate the possibility of interpreting the sunspot number time series as a non-stationary, multi-fractal and second-order fractal process.

The methodological basis of the paper includes the R/S method, the corrective function method, the wavelet transform modulus maxima method, and multifractal detrended fluctuation analysis. The R/S method is used both for constructing the corrective function and for analysing the sunspot number time series. The corrective function method, proposed as a special method for improving mono-fractal analysis accuracy, is used to compensate for nonlinear distortions in Hurst fractal dimension estimation (*Lazorenko et al., 2022*). The wavelet transform modulus maxima method and multifractal detrended fluctuation analysis are used to investigate the multi-fractal properties and non-stationarity of the analysed natural process (*Mallat, 1998*; *Kantelhardt et al., 2002*; *Chernogor et al., 2025*).

The theoretical significance of the study consists in further developing the methodological foundations of mono-fractal analysis and clarifying the limitations of the R/S method when it is applied to finite stochastic signals. The paper contributes to the understanding of how nonlinear estimation distortions arise and how they may be corrected by means of a specially constructed corrective function. It also contributes to the broader theory of fractal and multi-fractal analysis by linking mono-fractal Hurst dimension estimation with the study of non-stationary multi-fractal characteristics.

The practical significance of the results lies in the possibility of applying the corrective function method to improve the reliability of Hurst fractal dimension estimation in real physical, geophysical, solar-terrestrial and engineering signals. The proposed approach is particularly useful in cases where the numerical value of the fractal dimension has practical interpretative significance. The analysis of the sunspot number time series demonstrates that the method can be used for natural processes with complex scale-dependent behaviour, including those characterised by non-stationarity, multi-fractality and long-range dependence.

Thus, the present study is aimed at improving the accuracy and reliability of one of the classical methods of mono-fractal analysis. By combining the R/S method with the corrective function method and comparing the obtained results with multi-fractal analysis, the paper offers a methodological contribution to the investigation of complex natural processes and provides a practical example based on solar activity data.

Methods

R/S Method and Hurst Fractal Dimension

Well known, that many signals and processes in nature having as natural, as artificial origin have fractal properties and, therefore, can be classified as the physical (or natural) fractals. It is important to note, that in the most cases, such properties of the physical fractals exist in statistical sense only. In these conditions, it is reasonable to assume that these fractals must be described by some numerical characteristics having a statistical origin too.

Being well-known statistical characteristics, the Hurst exponent H been introduced by H. E. Hurst in 1951 (24 years before the fractals!) in the paper (*Hurst, 1951*) is simultaneously appeared to be the oldest statistical numerical characteristic from ones, which are used usually for fractals describing. In bounds of the so-called GBM Model proposed by B. Mandelbrot (*Mandelbrot, 1982*; *Feder, 1988*), the Hurst exponent H and fractal dimension D known as the Hurst fractal dimension are connected with the simple relation $D = 2 - H$.

It is important to point that in general case this relation is appeared to be incorrect (*Lazorenko & Chernogor, 2020*; *Lazorenko & Chernogor, 2023*). For example, for two-parametric fractal signal models, there is no dependence between the Hurst exponent H and the Hurst fractal dimension D at all. To explain this, seems, strange fact, it is necessary to consider the physical senses of these numerical characteristics (*Lazorenko & Chernogor, 2020*; *Lazorenko & Chernogor, 2023*). Having good localization in time domain, the Hurst fractal dimension D describes an existence of so-called SRD for a signal investigated. On contrary of this, for own correct estimation, the Hurst exponent H needs as long a signal as possible. As a result, its localization property is appeared to be poor, and the Hurst exponent H describes an existence of so-called LRD for a signal researched. Thus, there are the one-parametric fractal signal models, for which the SRD and the LRD are occurred to be the same (e.g., the GBM model, the FGN) model (*Lazorenko & Chernogor, 2023*). However, at the same time, for two-parametric fractal signal models, the SRD and the LRD are occurred to be different and independent (e.g., the Levi flight model, the Ornstein–Uhlenbek process model (FGN) model [16]). In this paper, we assume that all signals investigated exist in bounds of the GBM model and, therefore, the relation $D = 2 - H$ is assumed to be correct.

At present days, there are many different ways to estimate the Hurst exponent of a signal., The oldest and frequently the most reliable way to do this (*Feder, 1988*) is the Rescaled Range Analysis Method or simply the R/S method proposed by H. E. Hurst in 1965 (in the 'pre-fractal' era) (*Hurst et al., 1965*).

According to the new fractal analysis method classification introduced in 2023 (*Chernogor et al., 2023*), the R/S method having the statistical origin is appeared to be a member of the group of methods based on the statistical characteristics.

Let's consider the main idea of the R/S method. For a discrete signal s_i containing N points (the counter i varies in the bounds $i = \overline{1,N}$), its partial sums

$$y(n) = \sum_{i=1}^{n} s_i, \; n \ge 1,$$

the dispersion (the square of the standard deviations $S(n)$) given by the relation

$$S^2(n) = \frac{1}{n}\sum_{i=1}^{n}\left[s_i - \frac{1}{n} y(n) \right]^2, \; n \ge 1$$

and so-called 'range' $R(n)$ should be estimated. The range $R(n)$ of a signal s_i is given by the relation:

$$R(n) = \max_{0 \le t \le n}\left(y(t) - \frac{t}{n} y(n) \right) - \min_{0 \le t \le n}\left(y(t) - \frac{t}{n} y(n) \right).$$

Grounding on the values obtained, the so-called R/S statistics is built:

$$\frac{R}{S}(n) \equiv \frac{R(n)}{S(n)} = \frac{1}{S(n)}\left[\max_{0 \le t \le n}\left(y(t) - \frac{t}{n} y(n) \right) - \min_{0 \le t \le n}\left(y(t) - \frac{t}{n} y(n) \right) \right], \; n \ge 1.$$

In 1951 (*Hurst, 1951*), basing on the results of his own empirical investigations, H. E. Hurst found that the mathematical expectation of the R/S statistics has a power-law relationship with the size of the observation window length n as:

$$E\left[R / S(n) \right] \sim C n^H, \tag{1}$$

where C is some limited, positive constant, which doesn't depend on n, H is the Hurst exponent, $E[\,]$ is the operation of a mathematical expectation calculation.

Thus, changing the window length n, the dependence of the logarithms of $E[R/S(n)]$ vs. the logarithms of n can be obtained. The corresponding plot of this dependence should be built namely on the log-log plain. It is important to point, that if a signal s_i investigated has really the self-affine (and, therefore, fractal) properties, the experimental points should be grouped around some straight line. The Hurst exponent H is occurred to be equal to the angle coefficient of this straight line and can be simply estimated with usage of the least square method. On contrary, if that points were appeared to be not somehow grouped around any straight line, it can be claimed, that a signal investigated as whole hasn't a self-affine property and, therefore, cannot be considered a fractal one. Thus, for whole signal in such case, the Hurst exponent H cannot be estimated.

On the same time, in most practical cases, a signal researched is appeared to be fractal in some limited scale range only, not in all existing one. In such case, only some part of the pointes placed on the double logarithmic coordinates (on the log-log plain) in some scale range only can be successfully approximated with a straight line discussed above. Namely this straight line allows estimating the Hurst exponent H value for such limited scale range, and, therefore, one can speak about the fractal properties of the signal or process researched in this limited scale range. By the way, sometimes, two or even more such limited scales ranges are occurred to be exist on the same plot (*Feder, 1988*). Such signals are classified as the multi-fractals (the bi-fractals etc.), not the mono-fractals (*Lazorenko & Chernogor, 2023*).

Now some words about the valid values of the Hurst exponent are needed. As it had been discovered by B. Mandelbrot (*Mandelbrot, 1982*; *Feder, 1988*), for fractals, the Hurst exponent value H should be limited in the range $0 < H < 1$. Otherwise, a signal researched is occurred to be not self-affine and, therefore, is not fractal (*Feder, 1988*) even in the case, when the experimental points on the log-log plain have been excellently approximated by a straight line. Only if the condition $0 < H < 1$ has successfully satisfied, then one can speak about existence of some mono-fractal properties in the given scale range for the signal investigated.

Moreover, the Hurst exponent values are directly connected with so-called 'memory' effect for the signals researched. It is well known (*Mandelbrot, 1982*; *Falconer, 1990*; *Feder, 1988*), that if $0.5 < H < 1$, then a fractal signal has the property of the persistence, if $0 < H < 0.5$, then it has the property of the anti-persistence, if $H = 0.5$, then the 'memory' effect is occurred to absent and , thus, any tendency in the signal behavior is not saved. Namely the existence of this 'memory' effect leads to the LRD appearance described above.

It is important to note, that one can to apply the R/S method for investigations as in the time domain, as in the space one. In the second case, it is necessary simply to replace formally a time variable with a space one.

The last peculiarity of the R/S method application in this paper deals with an absence of any time resolution for the Hurst exponent, since it is estimated for the entire signal. Nevertheless, in many cases, the real processes in the nature, especially being in open, non-linear, dynamical systems (*Chernogor, 2008*), are occurred to be significantly non-stationary. Therefore, their fractal properties can significantly vary with time too. A solution of this problem is simple: the Hurst exponent H should be estimated with usage of some well localized (or even finite), window $W(t)$ sliding in the time domain. In such case, the Hurst exponent value will be estimated for some defined part of the signal investigated, but not for entire signal at once. Therefore, the Hurst exponent becomes a function of the time $H = H(t)$. No doubt, the minimum of the possible window width is defined by the statistical requirements to the sample length. It is necessary to point, that, on our opinion, it is convenient to associate each Hurst exponent value calculated with usage of the sliding window $W(t)$ in time domain with corresponding given time location of the window center.

In this paper, the R/S method are used as for the Corrective Function building, as for the sunspot numbers time series analyzing.

Corrective Function Method

In 2022 in the paper (*Lazorenko et al., 2022*) by L. F. Chernogor, O. V. Lazorenko and A. A. Onishchenko, so-called 'Corrective Function Method' (CF method) as a special method of the mono-fractal analysis has been proposed. The main idea of this universal method is in following.

Let's consider any mono-fractal analysis method, in which there is an estimation D^* of unknown fractal dimension D for a signal researched. This discrete signal s_i contains N points ($i = \overline{1, N}$). In general, the estimation D^* is an unknown non-linear function of D and N,

$D^* = f(D, N)$. In an ideal case, which doesn't exist on practice at all, this function should be linear and very simple, $D^* = D$, and should not be a function on N. Therefore, the estimation D^* obtained with using of the given mono-fractal analysis method, is occurred to be significantly distorted namely by this non-linear function $D^* = f(D, N)$. Understanding the reasons explained here, the idea of the CF method is appeared to be quite simple and clear.

The CF method purpose is neutralizing the influence of the function $D^* = f(D, N)$ non-linearity in some way and increasing an accuracy of the fractal dimension D estimation obtained with given mono-fractal analysis method. To do this, in the paper (*Lazorenko et al., 2022*), it was proposed an idea to inverse the non-linear function $D^* = f(D, N)$ on the base of the so-called 'Corrective Function' (CF). Using some test signal set, this CF should be constructed on the discrete grid over the (D, N) plain separately for each given mono-fractal analysis method. The test signal set should be based on the model fractal signals with known and controlled values of the variables D and N. The model fractal signals can be as deterministic and stochastic too. The steps of changing for D and N variables can be chosen in any reasonable way. Surely, the smaller, the better, but the time of the CF calculation matters significantly too.

Thus, when the CF on the discrete grid, $D^*_{ij} = f(D_i, N_j)$, $i = \overline{1, n}$, $j = \overline{1, m}$, has been ready, the inversion of the non-linear function $D^* = f(D, N)$ relatively first variable D should be started. Fixing N variable on the value $N = N_{sig}$, where N_{sig} is the length of the data vector containing the signal analyzed, we obtain the function $D^* = f(D, N_{sig})$ being a function of only one variable D on the interval $1 \le D < 2$. Well known, that an inverse function $D = f^{-1}(D^*, N_{sig})$ can exist on some interval only in the case, when the direct function $D^* = f(D, N_{sig})$ is monotonic there. Moreover, in our case, the function $D^* = f(D, N_{sig})$ should be a rising function of D on the interval $1 \le D < 2$.

Now let's return to the N_{sig} value. As a rule, for comparatively big values of N_{sig}, there are no problems with monotonicity of the function $D^* = f(D, N_{sig})$. But when N_{sig} value decreases, the $N_{\min}$ value, below of which ($N_{sig} < N_{\min}$) the monotonicity of the function $D^* = f(D, N_{sig})$ is occurred to be disrupted, appears. As well as all detailed explanations of this process appearance causes and the bulky relations for the fractal dimension D and its estimation error D are closely considered in the paper (*Lazorenko et al., 2022*), which, if needed, can be successfully downloaded for free, we avoid to repeat them here. Nevertheless, on our opinion, the algorithm of determination of the N_{sig} value is appeared to be very useful, since it allows to prepare a well-founded and reasonable answer on the question of what exactly is the minimum number of signal points $N_{\min}$ and why should be used in the given method of mono-fractal analysis. Before the appearance of the paper (*Lazorenko et al., 2022*), there was no such answer. For example, many years ago, in the book (*Feder, 1988*), the world famous fractalist J. Feder had claimed that for the R/S method, $N_{\min}$ value should be equal at least 2500. But in the paper (*Lazorenko et al., 2022*), the value $N_{\min}$ was shown to be principally much smaller ($N_{\min} = 32$) due to the reasons described above. Nevertheless, it is necessary to point, that if N_{sig} value decreases, the error of the Hurst fractal dimension estimation $\mathrm{D}D$, surely, rises.

In this paper, the CF method are used as for the Corrective Function building, as for the sunspot numbers time series analyzing.

Wavelet Transform Modulus Maxima Method

The WTMM method is based on the Continuous Wavelet Transform (*Mallat, 1998*). Being the basic informational characteristics of the multi-fractal analysis, the multi-fractal spectrum function $f(a)$ of the signal investigated is connected with the CWT SDF of the signal. The a value is known as the Holder exponent.

Suddenly, WTMM method has one significant disadvantage. It doesn't allow to consider the non-stationarity of the signal investigated as well as in this method the signal is investigated at once. At the same time, it is reasonable to predict that all multi-fractal characteristics of a non-stationary signal can significantly vary with time. Thus, it is necessary to apply another method, which is free from this disadvantage.

Multi-Fractal Detrended Fluctuation Analysis Method

The MFDFA method is appeared to be convenient to the non-stationary signal investigations in sliding time window (*Kantelhardt et al., 2002*).

The basic idea of the MFDFA method is following. Basing on the signal multi-fractal spectrum $F(a)$ analysis (the multi-fractal spectrum of the whole signal was denoted above as $f(a)$) and the sliding time window $W(t)$ application, the time dependences of location (minimal $a_{\min}(t)$ and maximal $a_{\max}(t)$ values of a) and of width ($\mathrm{D}a(t)$, $\mathrm{D}a = a_{\max} - a_{\min}$) of the multi-fractal spectrum can be obtained. Special attention should be paid to the location a^* of the multi-fractal spectrum maximum, given by the requirement $F(a^*) = \max_a F(a)$. The a^* value is called as the generalized Hurst exponent as well as for mono-fractal signal we have $\mathrm{D}a = 0$ and $a^* = H$. The generalized Hurst exponent a^* describes a multi-fractal support of the signal analyzed. Its fractal dimension is given by relation $D_a = 2 - a^*$.

In this paper, the WTMM method is used for investigation of the multi-fractal properties of the entire signal and the MFDFA method is applied to discover the non-stationarity of the multi-fractal properties of the signal and to compare these results with ones obtained with R/S method used with a sliding window in time domain. Namely such approach to the mono-fractal and multi-fractal analysis of a signal has been successfully used by the authors of this paper, for example, in (*Chernogor et al., 2025*).

Literature Review

Sunspot Numbers Time Series.

The time series of sunspot numbers (well known as the 'Wolf numbers' too), which characterizes solar activity and is associated with the counting of sunspots, is unique in the sense that observations have been conducted since 1610, although complete and reliable data have only been available since 1849 (*Vitinskii, 1965*). It is well known that it has a pronounced 11-year cycle, superimposed perhaps on another slower component with a period of about a hundred years (*Frame & Urry, 2016*). However, this information was obtained in the "pre-fractal" era. Surprisingly, the first fractal analysis of the time series of Wolf numbers was performed by B. Mandelbrot and J. Wallis in 1969 in the work (*Mandelbrot & Wallis, 1969*), i.e. six years before the appearance of the concept of "fractal". Been translated into current terminology, it turned out that in the ranges of periods from 3 to 30 months and from 30 to 100 years, the fractal dimension is $D \approx 1.1$. These results were obtained using the R/S method. In a number of later works quite similar results were obtained using other methods of fractal analysis, in particular, $D \approx 1.2$ in (*Ruzmaikin et al., 1994*), $D \approx 1.0 - 1.2$ in (*Ogurtsov, 2004*) and $D \approx 1.2 - 1.3$ in (Rypdal & Rypdal, 2012).

Thus, it has been established that the time series of sunspot numbers is a fractal process, well described by the model of generalized Brownian motion. The fractal dimension D of this process is in the range $D \approx 1.1 - 1.3$. Consequently, the series of sunspot numbers has the property of persistence.

Since 2006 (*Movahed et al., 2006*), regular studies of the multifractal properties of the Wolf number series have been conducted. It has been established that the sinusoidal trend has a significant negative impact on the estimates obtained. After its removal, it turned out that the fractal dimension of the series as a whole is $D = 1.88 \pm 0.01$ (*Movahed et al., 2006*). And this value differs significantly from that given by mono-fractal analysis methods. The explanation for this is simple: it is necessary to compare either the results obtained without a trend or with a trend. And this, unfortunately, was not done in (*Movahed et al., 2006*). Moreover, in 2009, in the work (*Hu et al., 2009*), a different mechanism for removing the sinusoidal trend was proposed, and the value for the fractal dimension turned out to be different: $D \approx 1.26$, which is quite consistent with the results presented above. Let us add that one of the first attempts to study the multifractal properties of the sunspot number series was made in 2005 in the work (*Lazorenko et al., 2005*). The refinement of individual aspects of the multifractal properties of the sunspot number series continues to this day (*Wu et al., 2015*).

Results

Corrective Function Bullt with Stochastic Mono-Fractal Signals Usage

Let's consider the non-linear function $D^* = f(D, N)$, which can be created for the R/S method on a test signal set. In this paper, the test signal set is based on the model stochastic mono-fractal signals. Such model choice is explained by the next reason. As a rule, the signals and the processes analyzed with the R/S method application are occurred to be fractal in statistical sense only (not in geometrical or algebraical ones). Therefore, it is reasonable, if the test signals are stochastic too, not deterministic. By the way, in the paper (*Lazorenko et al., 2022*), the test signals sets based on the model deterministic fractal signals were successfully used.

All model stochastic mono-fractal signals used in this paper are based on the modified cosine Weierstrass – Mandelbrot function (*Bandt et al., 2024*):

$$MW_D(t)=\sum_{n=0}^{\infty}\lambda^{(D-2)n}\cos\left(\lambda^{n}t+\varphi_n\right),$$

where λ is a numerical parameter ($\lambda > 1$), D is a fractal dimension ($1 \le D \le 2$), φ_n are the stochastic phases having some chosen distribution law at the interval $[0,2\pi]$, t is dimensionless time variable. At the (*Figure 1*), there are two examples of such model signal realizations with different fractal dimension D values. It is worth pointing that for the model signals we used the stochastic phases φ_n with normal (Gaussian) distribution law and a discrete grid over the (D,N) plain, where D value was changed in bounds $1 \le D \le 2$, with the step 0.001, and N value was given by the relation $N = 2^k$, $k \in \mathbb{N}$, $\mathbb{N}$ is the natural numbers set. Comparing with the paper (*Lazorenko et al., 2024*), in present case the step of D value changes was decreased ten times. Surely, this had requested much more time for calculations. Nevertheless, it was found that for most practical purposes, the step being equal to 0.01 is appeared to be quite sufficient and its decreasing seems to be no needed.

Due to the model fractal signals are stochastic, for the CF building, it is necessary to use many different realizations with the same combinations of D and N values with consequent averaging of the results obtained. In each case, the averaging has been provided over 300 different stochastic realizations of the model signals. For each combination of D and N values, as the estimation of the Hurst fractal dimension value D^*, as its error ΔD^* are calculated. In all calculations, the confidence level was assumed to be equal to 0.9.

At the (*Figure 2*), a comparison 'ideal' function $D^* = D$ (1) vs the non-linear function $D^* = f(D,N)$ having fixed N values: $N = 32$ (2), $N = 128$ (3), $N = 512$ (4) and $N = 2048$ (5) are shown. Discussed above non-linearity of this function is clearly shown. Three interesting tendencies in the $D^* = f(D,N)$ function behavior discovered in the paper (*Lazorenko et al., 2024*) have been successfully confirmed for the smaller step of the D value changes. First, for fixed D value, the difference between D^* and D decreases with N value raising. Second, the maximal error of the D^* value estimation doesn't exceed approximately 3%. Third, the non-linearity of the $D^* = f(D,N)$ function leads to the following. Depending on the N value, there is some bound value D_0, for which at the interval $1.0 \le D \le D_0$, the Hurst fractal dimension estimations D^* have the tendency to be overestimated, but at the interval $D_0 \le D \le 2.0$ – on the contrary, to be underestimated. So, for $N = 32 - 2048$, the D_0 value is found to be slightly increasing in bounds $1.3 < D_0 < 1.4$ with N value rising.

The small gray rectangle on the (*Figure 2*) is corresponding to some well-known 'strange' results obtained by H. E. Hurst with the R/S method application in the 'pre-fractal' era yet. He found (*Feder, 1988*; *Hurst et al., 1965*) that the Hurst exponent H is more or less symmetrically distributed about a mean of 0.73, with a standard deviation of about 0.09. In this case, the Hurst fractal dimension D has a mean of 1.27 and the same standard deviation, $D = 1.27 \pm 0.09$. In the paper (*Lazorenko et al., 2024*), a possible explanation of this fact has been proposed. As well as this paper is download free, we shall not retell its contents and should be limited only by main idea of this explanation listed below in 'Discussion'. It is necessary to note that the results obtained in this paper has well agreed with this idea.

Thus, for any researches provided with the R/S method usage the CF method application is strictly recommended.

Mono-Fractal and Multi-Fractal Analysis of the Sunspot Numbers Time Series

To demonstrate the results of simultaneously application of the R/S method and the CF method for investigations of a real natural physical process, the sunspot numbers time series analysis is considered.

For present analysis, the daily total sunspot numbers from 1/1/1818 till 31/07/2024 were chosen (*SILSO World Data Center, n.d.*). Of course, as well as all useful information about fractal structure of a signal is consisted in high-frequency part of Fourier spectrum, any averaging is not preferable. But the low-frequency part impact on the Hurst dimension estimations can be successfully reduced by differentiation operation usage. Namely this approach is used in our investigations.

As it had been found by B. Mandelbrot (*Mandelbrot & Wallis, 1969*), the sunspot time series in whole is not mono-fractal. However, there are some ranges of periods, where mono-fractal properties can be discovered. From the (*Figure 3*), where the dependence of $\log(R / S)$ vs $\log n$ given by the relation (1) is shown, such ranges and corresponding values of Hurst fractal dimension D are defined. It was found that in the ranges of periods from 7 to 70 days $(n = 1.95 - 4.25)$ $D = 1.66 \pm 0.01$, from 3 to 30 months $(n = 4.50 - 6.80)$ $D = 1.91 \pm 0.01$, from 30 to 90 months $(n = 6.80 - 7.90)$ and from 30 to 200 years $(n = 9.29 - 11.20)$ $D = 1.58 \pm 0.01$. In other ranges, where given dependence cannot be successfully approximated by a straight line, the sunspot time series is appeared to be not fractal. If for straight line approximation building, one uses all points shown at the (*Figure 3*), some averaged Hurst fractal dimension D value, which is able to characterize the whole signal investigated. This value is found to be $D = 1.78 \pm 0.01$.

At the same time, it well known (*Mandelbrot, 1999*; *Harte, 2001*), that a fractal, which has mono-fractal properties at least in two different ranges of periods, is appeared to be a multi-fractal. Therefore, it is worth investigating with application of special multi-fractal methods. The most popular from them are the WTMM method and the Multi-Fractal Detrended Fluctuation Analysis described above in this paper. First of them is suitable for investigations of multi-fractal properties of whole signal and second of them—for estimations of multi-fractal characteristics in sliding window in time domain.

In bounds of the WTMM method, it was found that the multi-fractal characteristics of the whole sunspot time series are following: $a_{\min} = - 0.02$, $a_{\max} = 2.21$, $\mathrm{D}a = a_{\max} - a_{\max} = 2.23$, $a^{*} = 0.28$ $\alpha^{*} = 0.28$. The corresponding multi-fractal spectrum is shown at the Figure 4. A good agreement between the obtained values of the Hurst fractal dimension D » 1.78 and the fractal dimension $D_{a} = 2 - a^{*}$ » 1.72 is found. Moreover, it is suitable to suppose that both the properties mono-fractal and multi-fractal can be non-stationary in sense of their numerical characteristics should be been estimated in sliding window in time domain. This is fully realized as with R/S method for $H(t)$ (*Figure 5f*), as with MFDFA for $a_{\min}(t)$, $a_{\max}(t)$, $\Delta a(t)$ and $a^*(t)$ (*Figure 5b–e*). Here the horizontal straight lines denote the corresponding values obtained for the whole signal with WTMM method. It is necessary to point that all these time functions were smoothed with the window, which had the width been equal to 1/20 of all function length in time domain. There is a good agreement between time variations of $H(t)$ (*Figure 5f*) and $a^*(t)$ (*Figure 5e*). All calculations were performed with usage of time domain window with width $T_w = 11$ years, which corresponds with well-known period existing in time-frequency structure of the sunspot number series (*Figure 5g*). On the (*Figure 5g*), the existence of the tree-like structure confirms well the presents of fractal properties for the sunspot numbers series.

In 2023 in the article (*Chernogor et al., 2023*), the concept of the so-called 'second-order fractals' has been introduced. By the definition, second-order fractal is a fractal, fractal dimension of which is appeared to be a fractal function of time or space variable. A fractal signal with such property is called as second-order fractal signal. If the functions $H(t)$ and $a^*(t)$ are considered without smoothing (instead of that was done at the Figure 5), they were found to be namely a second-order fractal signals and, therefore, the sunspot time series was occurred to be a second-order fractal process too. The Hurst fractal dimensions of the functions $H(t)$ and $a^*(t)$ are $D = 1.68 \pm 0.01$ and $D = 1.53 \pm 0.01$ correspondently.

Discussion

With each new day, the methods of mono-fractal and multi-fractal analysis are occurred to be in demand more and more in different branches of the science, the engineering and the technologies. The fractal numerical characteristics, which produced by different analysis methods, have not a pure theoretical value only, but become a huge practical significance. At the same time, there are no 'ideal' estimators for these numerical characteristics. Each analysis method has a personal own set of as advantages, as disadvantages. Therefore, the problem of increasing the accuracy of the fractal numerical characteristics estimations is appeared to be valuable and important.

One of the possible ways this problem solving is the CF method application for any mono-fractal and multi-fractal analysis method. In this paper, this was successful performed on the example of the R/S method being one of the oldest mono-fractal analysis methods.

Surely, the CF method has some disadvantages too. First, at the time, there is no clear recommendation regarding test model sets. Each researcher can create his own fractal models as deterministic, as stochastic. And the CF created with this test signal set will be depend from this set. Second, the process of the CF building requires as many time, as big computing power. May be, in future, it would be advisable to combine the efforts of all researchers and, e.g., to create a corresponding library in the Internet.

It is surprisingly, but CF method appearance has caused, at least, one unexpected result. In the article (*Lazorenko et al., 2024*), a hypothesis trying to explain well-known 'strange' results obtained by H. E. Hurst with the R/S method application in the 'pre-fractal' era has occurred. On opinion of the authors of the article (*Lazorenko et al., 2024*), an existence of the significant shift for the Hurst fractal dimension D^* estimations observed for persistent natural physical processes can be explained rather by special features of the R/S method applied for processing of the experimental data, than by own really existing properties of these processes. Of course. This is a hypothesis only, but, at least, it tries to find an answer on the question being unanswered at all many times.

Conclusions

1. The R/S analysis method is the oldest and the most popular way to estimate a Hurst exponent for any signal or process. In bounds of the GBM model, the Hurst fractal dimension D and the Hurst exponent H are connected with a simple relation $D = 2 - H$.
2. Grounding on the results of numerical modelling with simultaneous usage of the CF method and the set of model stochastic mono-fractal signals based on the modified cosine Weierstrass – Mandelbrot function, for R/S method, it was found that the dependence between a fractal dimension value D^* estimation and a true own fractal dimension value D is appeared to be principally non-linear.
3. The main peculiarities of the R/S method as the most popular estimator of Hurst fractal dimension were investigated. The corresponding corrective function was built. At the example of the mono-fractal analysis of the sunspot time series, the possibilities of the simultaneous application of the R/S method and CF method are shown.
4. The results obtained with mono-fractal analysis were compared with the results of multi-fractal analysis as for whole signal (WTMM method), as in sliding time-domain window (MFDFA method). The set of mono-fractal and multi-fractal characteristics values for the sunspot time series were successfully specified and corrected.
5. The sunspot time series was found to be second-order fractal process.

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Conflict of Interest

The authors declare that there is no conflict of interest.

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Appendix

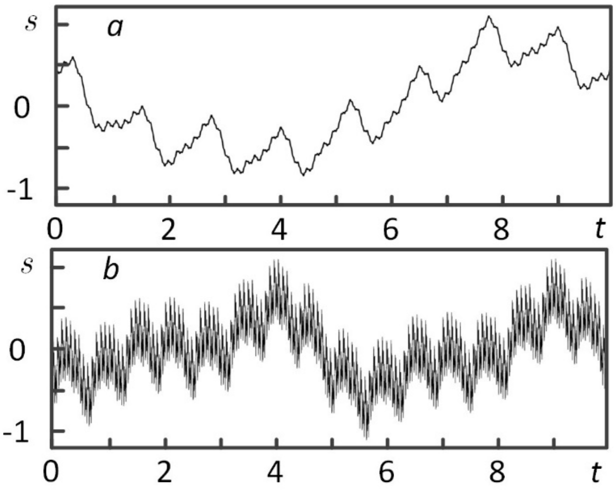

Figure 1. Model stochastic mono-fractal signals with different fractal dimension values: D=1.2 (a) and (b).

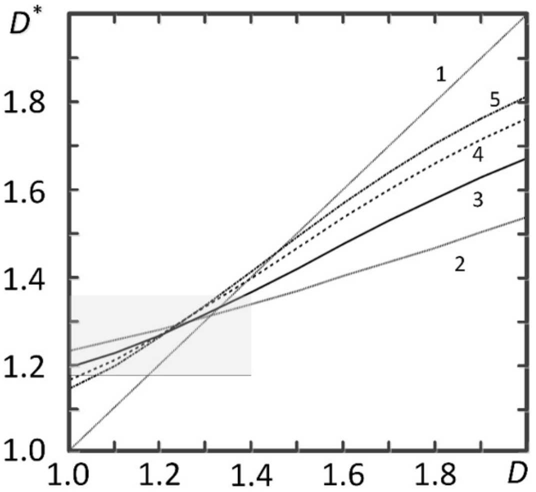

Figure 2. Comparison of the 'ideal' function $D^* = D$ (1) vs the non-linear function $D^* = f(D, N)$ having fixed N values: $N = 32$ (2), $N = 128$ (3), $N = 512$ (4) and $N = 2048$ (5). The gray rectangle denotes a graphical view of the 'strange' result obtained by H. E. Hurst.

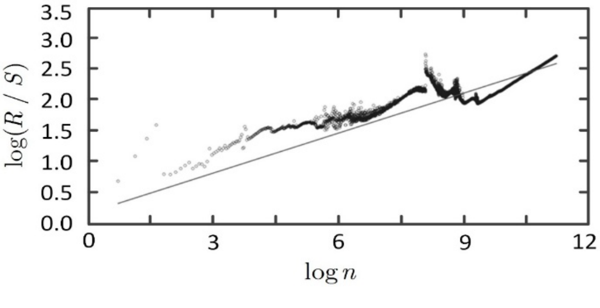

Figure 3. The dependence of $\log(R / S)$ vs $\log n$ defined by (1) and obtained with the rescaled range analysis method application for the sunspot numbers time series

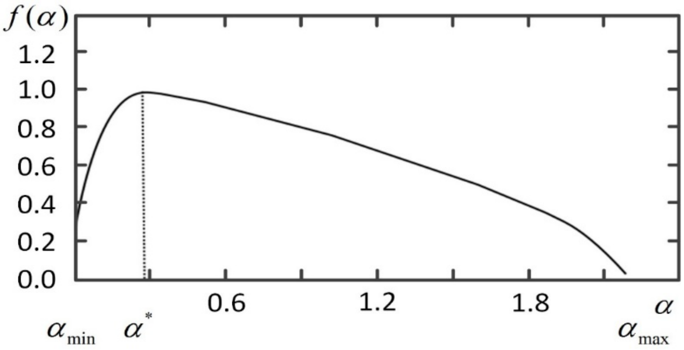

Figure 4. The multi-fractal spectrum function $f(a)$ obtained with the wavelet transform modulus maxima method application for the whole sunspot numbers time series

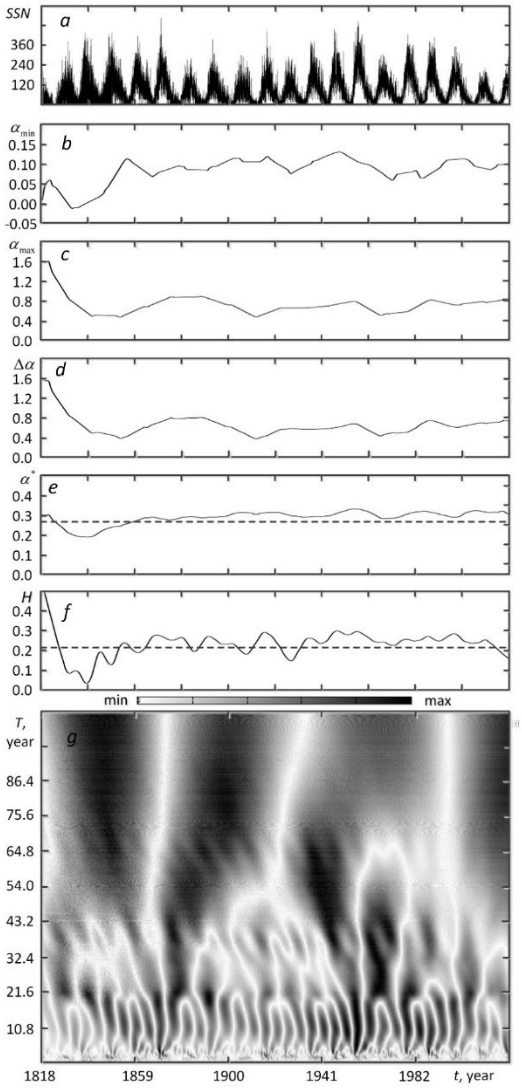

Figure 5. The results of the non-stationary fractal structure of the sunspot number time series from from 1/1/1818 till 31/07/2024 with usage of the R/S method with sliding window and the MFDFA method: a – the sunspot time series in time domain, b – $a_{\min}(t)$, c – $a_{\max}(t)$, d – $\mathrm{D}a(t)$, e – $a^{*}(t)$, f – $H(t)$, g – the continuous wavelet transform spectral function module $\left|W(T,t)\right|$ obtained with usage of the fourth order Daubechie's wavelet